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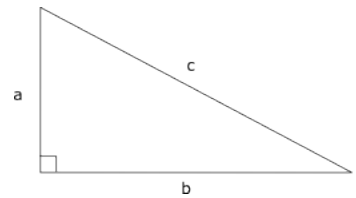
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 10, July 25, 2016

1. Since each 3×4 and 4×3 rectangle needs to have at least one black square, the minimum possible is 12. This can be achieved with the following configuration.

		X				X			X		
					X						
		X							X		
					X			X			
			X								
		X				X			X		

2. By substituting the two points $A(-2; 1)$ and $B(2; 9)$ into the equation of the parabola,



4. Let a and b be the shorter two sides of the triangle and c be the hypotenuse. Then we have
$$\frac{1}{2}ab = 3(a + b + c).$$
Dividing both sides by 3, using $c = \sqrt{a^2 + b^2}$ and rearranging
$$\frac{ab}{6} - (a + b) = \sqrt{a^2 + b^2}.$$
Squaring both sides,
$$\frac{a^2b^2}{36} - \frac{ab}{3}(a + b) + (a + b)^2 = a^2 + b^2.$$
Simplifying,
$$a^2b^2 - 12ab(a + b) + 72ab = 0.$$
Factoring ab,
$$ab(ab - 12(a + b) + 72) = 0.$$
We know that $ab \neq 0$ so we are left with
$$ab - 12(a + b) + 72 = 0.$$
Now using the hint we get
$$(a - 12)(b - 12) = 72 = 2^3 \cdot 3^2.$$
By equating all possible factorisations of 72, we get 6 pairs of solutions for a and b as follows
$$(13, 84), (14, 48), (15, 36), (16, 30), (18, 24), (20, 21).$$

5. Using the difference of two cubes, we get
$$(x - y)(x^2 + xy + y^2) = 91 = 1 \times 91 = 7 \times 13.$$
Also notice that $x^2 + xy + y^2 = (x + y/2)^2 + 3y^2/4 \geq 0$ which also tells us $x - y \geq 0$. Since 7 and 13 are prime, by equating the factors on the LHS with ways to factorise 91 we have 4 cases:

   (a) $x - y = 91$ and $x^2 + xy + y^2 = 1$

   (b) $x - y = 1$ and $x^2 + xy + y^2 = 91$

   (c) $x - y = 13$ and $x^2 + xy + y^2 = 13$

(d) $x \quad y = 7$ and $x^2 + xy + y^2 = 7$

The rst and third cases have no solutions, the second case has solutions $fx = 5; y = 6g; fx = \quad 6; y = \quad 5g$ and fourth case has solutions $fx = 3; y = \quad 4g; fx = \quad 4; y = 3g$.

6.

Senior Questions

1. For $p = 2$ we have $2^2 + 2^2 = 8$ which is not prime. For $p = 3$, we have $2^3 + 3^2 = 17$ which is prime. For $p > 3$ (odd), we claim that $2^p + p$