MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 14, August 22, 2016

1. In $\triangle ABC$, let $\angle CAB = 60$. The perpendicular from the midpoint of side AB intersects the extension of AC at the point N. The perpendicular from the midpoint of side AC intersects the extension of AB at the point M. Prove that $CB = MN$.

2. Let x, y and z be integers. Prove that if $2x + 4y + 5z$ is a multiple of 17, then so is $3x + 6y - z$.

3. Consider an $n \times n$ table. In each square of its first column someone has written the number 1, in each square of the second column, number 2, and so on. Then someone erased the numbers on the diagonal which connects top-left with bottom-right angle of the table. Prove that the sum of the numbers above the diagonal is twice the sum of the numbers under it.

4. Let p, q, r and s be real numbers such that $p^2 + q^2 = 1$ and $r^2 + s^2 = 1$. Prove that

$$(pr + qs)^2 \leq 1.$$

5. Several diagonals (possibly intersecting each other) are drawn in a convex n-gon in such a way that no three diagonals intersect in one point. If the n-gon is cut into triangles, what is the maximum possible number of theses triangles?

6. Let

$$x = \frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \frac{1}{9999} + \frac{1}{99999} + \dots,$$

If x is written as a decimal, find the 17th digit after the decimal point; also, find the 59th digit after the decimal point.

Senior Questions

1. Nancy shuffles a deck of 52 cards and spreads the cards out in a circle face up, leaving one spot empty. Andy, who is in another room and does not see the cards, names a card. If this card is adjacent to the empty spot, Nancy moves the card to the empty spot, without telling Andy; otherwise nothing happens. Then Andy names another card and so on, as many times as he likes, until he says \stop."

   (a) Can Andy guarantee that after he says \stop," no card is in its initial spot?

   (b) Can Andy guarantee that after he says \stop," the Queen of Spades is not adjacent to the empty spot?

2. A circle on diameter AB is given, together with a point P inside the circle but not on AB. Show how to construct, using only an unmarked string, a line through P perpendicular to AB. Prove that your construction succeeds.

3. Let x and y be positive integers, and let p be any odd prime. Suppose n is a positive integer which contains an odd factor greater than one, prove that $x^n + y^n = p$ has no solutions.

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