MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 10, July 29, 2014[1]

1. To find the expected value we must sum $\sum sp_s$ where s

Each new throw is independent of the last, so the expected score of 3 throws is thrice the expected score of one, i.e. $3s = 37{:}3818$.

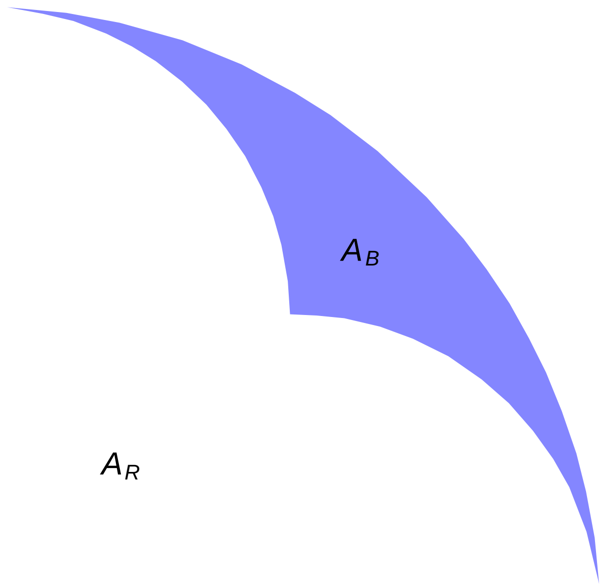

Figure 1: Picture for question 2

2. Let the side of the square be r, then $\frac{1}{4}\ \ r^2$ is the area of the larger quarter circle. So

$$
\begin{aligned}
\frac{1}{4}\ \ r^2 &= A_B + 2\frac{1}{2}\ \ \ \frac{r}{2}^{\ 2}\ \ \ A_R \\
&= A_B + \frac{1}{2}\ \ r^2\ \ \ A_R \\
0 &= A_B\ \ \ A_R \\
A_R &= A_B
\end{aligned}
$$

so the blue and red regions' areas are in ratio 1 : 1.

3. The rst number in the lexicographic ordering of arrangements of the digits 0 through 9 is 0123456789. If we want to nd the millionth we need to take 1 000 000 1 = 999 999 steps down the ordering. If we x the rst digit, there are 9! ways of arranging the remaining 9. So we want to nd how many lots of 9! we need to step to get at least to 999 999, so we should take the smallest integer so that 999 999 n 9! < 0, which means n = d999 999=9!e = 3. The third digit in our list is 2, so the millionth number must start with a 2. Now the numbers 2$:::$ start at the 2 9! + 1th spot, so we have 1 000 000 2 9! 1 = 274 239 spots to make up with lots of 8! (xing the rst two digits gives us 8! ways of arranging the rest), so d274 239=8!e = 7. The seventh number in our list (remembering that 2 has already been used) is 7, so we're up to 27$:::$. Continuing:

Number	Spots to make up	n
$27\ldots$	$1\,000\,000 - (2 \times 9! + 6 \times 8! + 1) = 32\,320$	$\lceil 32\,320/7! \rceil = 7$
$278\ldots$	$1\,000\,000 - (2 \times 9! + 6 \times 8! + 6 \times 7! + 1) = 2\,079$	$\lceil 2\,079/6! \rceil = 3$
$2783\ldots$	$1\,000\,000 - (2 \times 9! + 6 \times 8! + 6 \times 7! + 2 \times 6! + 1) = 639$	$\lceil 639/5! \rceil = 6$
$27839\ldots$	39	$\lceil 39/4! \rceil = 2$
$278391\ldots$	15	$\lceil 15/3! \rceil = 3$
$2783915\ldots$	3	$\lceil 3/2! \rceil = 2$
$27839154\ldots$	1	

So 2783915406 is the 999 999th number in the list because it is the first number starting with $27839154\ldots$, we have one more spot to make up, so the millionth number is 2783915460.

4.

5.

6. Note that $792 = \operatorname{lcm}(88, 99)$. Then

$$\left((88!)^{\frac{1}{88}}\right)^{792} = (88!)^9$$
$$\left((99!)^{\frac{1}{99}}\right)^{792} = (99!)^8.$$

Using this, let's take

$$\begin{aligned} \frac{(99!)^8}{(88!)^9} &= \left(\frac{99!}{88!}\right)^8 \frac{1}{88!} \\ &= \frac{(99 \cdot 98 \cdots 89)^8}{88 \cdot 87 \cdots 2 \cdot 1} \\ &= \frac{(99 \cdot 98 \cdots 89) \cdot (99 \cdots 89) \cdots (99 \cdots 89)}{88 \cdot 87 \cdots 2 \cdot 1}. \end{aligned}$$

Now looking at the above fraction, the top has $8 \cdot (99 - 89 + 1) = 88$ numbers multiplied together, as does the bottom, only all the numbers on in the numerator are larger than all the numbers in the denominator, so this fraction must be greater than 1. Then

$$(99!)^8 > (88!)^9.$$

Taking the 792nd root of both sides is ok because both numbers are greater than zero, so

$$\left((99!)^8\right)^{\frac{1}{792}} > \left((88!)^9\right)^{\frac{1}{792}}$$

or rather

$$\sqrt[99]{99!} > \sqrt[88]{88!}.$$