

MATHEMATICS ENRICHMENT CLUB.
 Solution Sheet 2, May 14, 2013

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1. $\text{LCM}(10; 12) = 2^2 \cdot 3 \cdot 5 = 60$ minutes.

2. $6528(10a + 3) = 8256(30 + a)$ implies $a = 4$

3. 3 play only the piano.

4. Let $p(x) = (3 + 2x + x^2)^{1998} = a_0 + a_1x + a_2x^2 + \dots + (0) = 1998(3+2 \cdot 0+0^2)^{1997} (2+2 \cdot 0) = 1998 \cdot 2 = 3996$

(b) $a_0 + a_1 + a_2 + \dots = p(1) = (3 + 2 + 1)^{1998} = 6^{1998}$

(c) $a_0 - a_1 + a_2 - \dots = p(-1) = (3 - 2 + 1)^{1998} = 2^{1998}$.

5. (a) Since $a + b + c = 2$ and $a + b > c$, $a + c > b$ and $b + c > a$ each;
 $1 + ab + bc + ca - abc > 0$

and

$$\begin{aligned} (a + b + c)^2 &= 4 \\ a^2 + b^2 + c^2 + 2(ab + bc + ca) &= 4 \\ ab + bc + ca &= 2 \end{aligned} \quad 1$$

$$\frac{1}{2}(a^2 + b^2 + c^2):$$

Combining the two yields the answer.

6. (a) Using the triangle inequality gives $AC < AB + BC$, $AC < AD + DC$, $BD < AD + AB$ and $BD < BC + CD$, summing all of these together gives

$$\begin{aligned} 2(AC + BD) &< 2(AB + BC + CD + AD) \\ AC + BD &< p: \end{aligned}$$

¹Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.

Mark as E the intersection of AC and BD, then again using the triangle inequality we have $AB < AE + EB$, $BC < EB + EC$, $CD < CE + ED$ and $AD < ED + AE$. Again summing all of these together gives

$$AB + BC + CD + AD < AE + EC + BE + ED + BE + ED + CE + EA$$

$$p < AC + BD + BE + CA$$

$$p < 2(AC + BD)$$

$$\frac{1}{2}p < AC + BD:$$

(b) The lines AE, BE, CE and DE divide the quadrilateral into 4 pieces. Say $\angle AEB = \alpha$, and $\angle BEC = \beta$, then by opposite angles $\angle CED = \alpha$ and $\angle AED = \beta$. The 4 angles must sum to 2π so $2\alpha + 2\beta = 2\pi \Rightarrow \alpha + \beta = \pi$. Note also that $\sin \alpha = \sin(\pi - \alpha)$. Now we may sum the area of the 4 triangles to determine the area of the quadrilateral:

$$a = \frac{1}{2}AE \cdot BE \sin \alpha + \frac{1}{2}BE \cdot CE \sin \beta + \frac{1}{2}CE \cdot DE \sin \alpha + \frac{1}{2}DE \cdot AE \sin \beta$$

A

O₁

O₂

P

C

B

O₃