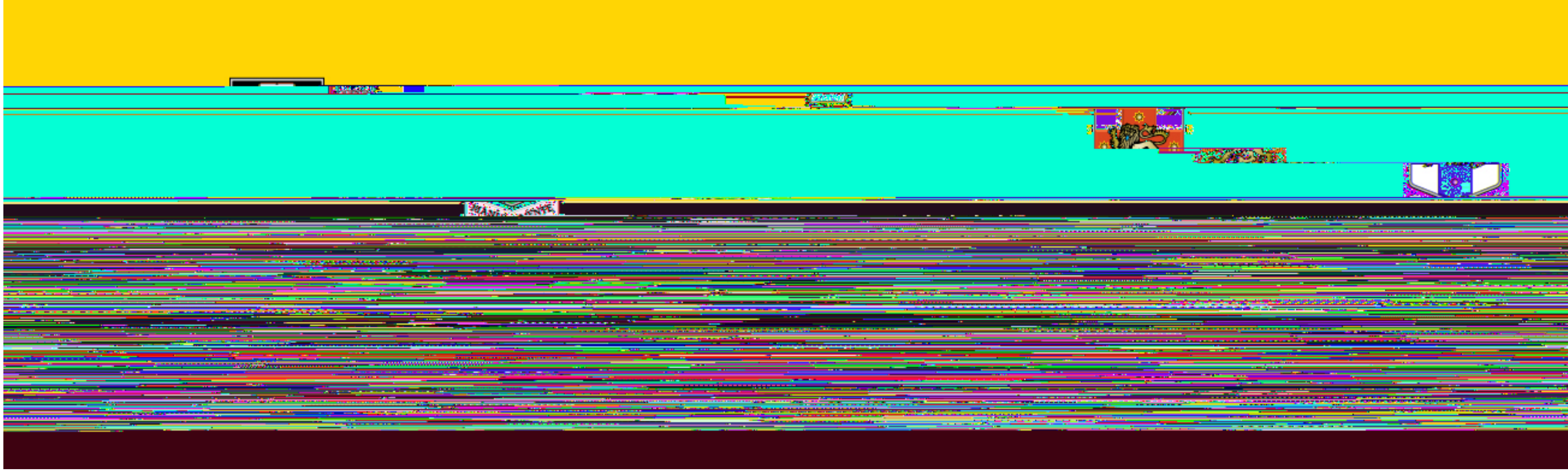

MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 16, September 10, 2013

1. The four digit numbers that satisfy the first equation are 1064, 1164, 1264 and so on. Of these, only 1764, 3364, 8464 are square. These satisfy the first equation for $n = 17$, 33 and 84 respectively. Then $201 \times 84 + 64 = 16\,948$ which is larger than 4 digits, $201 \times 33 + 64 = 6697$ which isn't square. So $n = 17$.

2. The result is clear if the line drawn is either of the diagonals, for they are lines of symmetry. Suppose now that the line drawn is not a diagonal. With the diagonals also drawn, the parallelogram is divided into 6 triangles. We'll leave the rest up to you, but these 6 triangles come in 3 pairs of congruent triangles, each having one of the pair on either side of the line, meaning that the line divides the parallelogram into two equal areas.

3. (a) Let A and B be the two vertices of our convex polygon that are the most distant from each other. Should we now draw two lines, one through A and one through B, that are perpendicular to AB we see that the polygon is contained inside this "strip". Now draw a further two lines, parallel to AB, that intersect the polygon only once (at a vertex), or run along an edge of the polygon (are tangent). We have now drawn a rectangle which encloses the polygon. Let C and D be two points at which these two lines, which are parallel to AB, intersect the polygon. The area of this rectangle is then $2 \times \text{Area}(\triangle ACB) + 2 \times \text{Area}(\triangle ADB)$. Since our polygon is convex, these two triangles must be wholly contained inside it, and so this sum is at most 2.

   (b) Continuing from above, AB must be on the sides of the triangle, meaning C is the third vertex, and D lies on AB. Any constructed parallelogram then, must have length AB and a height which is the same as that of the triangle ABC, and so area 2.

4. (a) A square number can either be the square of an even or an odd integer. Consider first $(2n)^2 = 4n^2$, whose remainder when divided by 4 is 0. Now consider $(2n+1)^2 = 4n^2 + 4n + 1$, which, when divided by 4 leaves a remainder of 1.

   (b) Suppose we have $(2n+1)$, $(2\ell+1)$ and $(2m+1)$ such that $(2n+1)+(2\ell+1)$, $(2n+1)+(2m+1)$ and $(2m+1)+(2\ell+1)$ are all square. This means $2(n$

$2(m + n + 1)$ and $2(m + \ell + 1)$ are all square. We must ensure these have a remainder 1 or 0 when divided by 4. Beacuse they're all even, they must be divisible by 4, which means $n + \ell + 1$, $m + n + 1$ and $\ell + m + 1$ are all even, but this cannot be the case. Suppose n is even and ℓ odd, so that $n + \ell + 1$ is even. Then to ahve $m + n + 1$ even, we must have m odd, but then $\ell + m + 1$ is odd.

Suppose instead we have 2ℓ, $2m + 1$ and $2n + 1$. Then the sums are $2(\ell + m) + 1$, $2(\ell + n) + 1$ and $2(m + n + 1)$. So we now need to ensure that $\ell + m$ and $\ell + n$ are even (to leave a remainder of 1 when dividing $2(\ell + m) + 1$ or $2(\ell + n) + 1$ by 4) and $m + n + 1$ is even (to leave a remainder of 0 when dividing $2(m + n + 1)$ by 4). If ℓ is odd, then m and n must both also be odd, leaving $m + n + 1$ odd, but if ℓ is even, then m and n must both be even, still elaving $m + n + 1$ odd. So it cannot be done.

Finally, if we have $2m + 1$, 2ℓ, and $2n$ then if ℓ and n are both even or both odd, $2(\ell + n)$ is divisible by 4, and if we also have m the same, then $2(m + \ell) + 1$ and $2(m + n) + 1$ have remainder 1 when divided by 4.

It also clearly works if all 3 numbers are even, $2n$, $2m$ and 2ℓ provided m, n and ℓ are all even or all odd.

(c) Try 19, 30 and 6.

5. (a) Suppose we number the land mass 1 through n. We can then write our walk as a sequence of numbers, e.g. $1 \to 2 \to 1 \to 3 \to 4 \to 5$. Assume we only cross each bridge exactly once, meaning in our example the $1 \to 2$ bridge is different from the $2 \to 1$ bridge. Excluding the beginning and ends of our walk, if we wish to pass through a land mass i and not end there must be 2 bridges per visit to i, thus an even number.

   (b) All the land masses in Königsberg have and odd number of bridges attached to them, so we cannot make a walk that crosses each bridge exactly once.

   (c) We can accomodate the lazy tourists by building one extra bridge. There are 4 land masses, each with an odd number of bridges attached to them. To have a lazy walk we are allowed to have 2 land masses with an odd number of bridges, or have every land mass have an even number of bridges. By building a bridge between any two land masses, we make those two land masses now have an even number of bridges, and the other two to have an odd number.

6. The maximum number of Knights that can be placed an a chessboard so that no two Knights attack each other is 32. If we place all the Knights on dark squares, then every light square can be attacked, so every square is either taken up by a Knight, or can be attacked.

   I managed to get 18 Knights on the board that only attack one other Knight, could you get more?