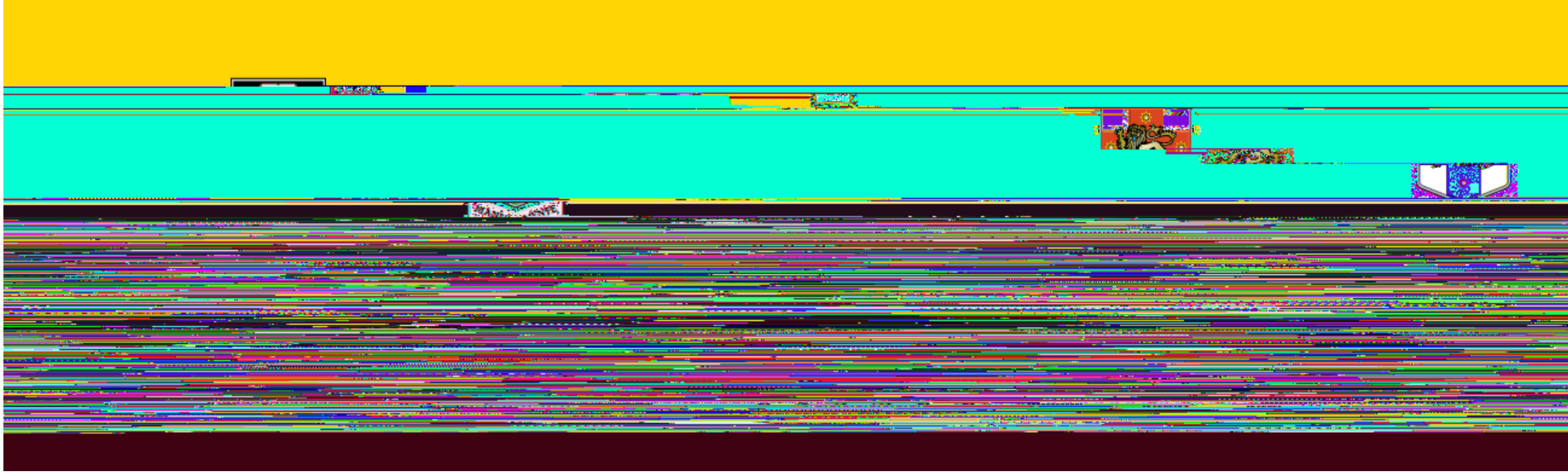

MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 13, August 20, 2013

1. Counting from right to left, there are 9 possible digits for the units columnn, 9 for the tens and 9 for the hundreds. If all of these are 0, then the only choice for the thousands is 3, otherwise the only choice is 2 - so there's only 1 choice for the thousands column. So there are $9 \times 9 \times 9 = 729$ years with no 1.

2. Suppose the score of the $i$th student is $s_i$, then we have $\frac{1}{20}\sum_{i=1}^{20} s_i = 66$ and $\frac{1}{30}\sum_{i=21}^{50} s_i = 56$. The total average is

$$\frac{1}{50}\sum_{i=1}^{50} s_i = \frac{1}{50}(66 \times 20 + 56 \times 30) = \frac{2}{5}66 + \frac{3}{5}56 = \frac{132 + 168}{5} = 60$$

so the average score is 60%.

3. (a)

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $x_n$ | 1 | 2 | 5 | 12 | 29 | 70 |
| $y_n$ | 1 | 3 | 7 | 17 | 41 | 99 |
| $\frac{y_n}{x_n}$ | 1 | 1.5 | 1.4 | 1.4167 | 1.4138 | 1.4143 |

(b) First note that

$$\begin{aligned} y_n^2 - 2x_n^2 &= 4x_{n-1}^2 + 4x_{n-1}y_{n-1} + y_{n-1}^2 - 2(x_{n-1}^2 + 2x_{n-1}y_{n-1} + y_{n-1}^2) \\ &= 2x_{n-1}^2 - y_{n-1}^2 \\ &= -(y_{n-1}^2 - 2x_{n-1}^2): \end{aligned}$$

Repeating this we see

$$\begin{aligned} y_n^2 - 2x_n^2 &= -(-(y_{n-2}^2 - 2x_{n-2}^2)) \\ &= -(-(-(y_{n-3}^2 - 2x_{n-3}^2))) \\ &\vdots \quad n-1 \text{ times} \\ &= (-1)^{n-1}(y_1 - 2x_1) = ( \end{aligned}$$

(c) Diving the above by $x_n^2$ we see that

$$\frac{y_n^2}{x_n^2} = 2 + \frac{(-1)^n}{x_n^2}$$

and as $x_n$ is increasing, the second term approaches zero as $n$ grows to infinity. So

$$\frac{y_n^2}{x_n^2} \to 2.$$

4. Suppose $ABC$ is the triangle, and $M, N$ the midpoints on $AB$ and $AC$ respectively, so that $CM$ and $BN$ are equal length. Since medians cut the area of a triangle in half, the areas $BMC$ and $BNC$ are equal (both half of the area of $ABC$). This implies that $\frac{1}{2}|CM||BC| \sin \angle MCB = \frac{1}{2}|BN||CB| \sin \angle NBC$

configurations when we consider equal all configurations that can be obtained using the rules provided. For example, suppose there are 4 Knights who sat $(1, 2, 3, 4)$ on the first day, then $(1, 3, 2, 4)$ is the same as $(3, 1, 4, 2)$ (we can swap 1 with 3, and 2 with 4).

We now define a "winding" number. For a given seating, Merling starts at Knight 1, then walks clockwise around the table until he reaches Knight 2, then continues clockwise until reaching Knight 3, and so on until he reaches Knight $n$, then the winding number of this seating is the number of times Merlin walked around the table. So, $(1, 3, 2, 4)$ has a winding number of 2 (he passes 1 and 2 on the first lap, then 3 and 4 on the second). Note that $(3, 1, 4,$