An insurance company faces a risk X over a period.

Reinsurance:

- $f(X)$! reinsurer
- $R_f(X) = X \quad f(X)$! insurer
- The insurer pays premium $(f(X))$ to the reinsurer
- Total risk exposure $S^f(X) = X \quad f(X) + \quad (f(X))$

To minimize

$$(S^f(X))$$

Three factors

- the optimization objective on $S^f(X)$
- is a the premium principle
- f is the ceded loss function

- Value-at-Risk: $\mathrm{VaR}\ (X) = (\ F_X)_L^{\ 1}(\ )$ (Solvency II);
- Expected Shortfall (ES): (Swiss Solvency Test) For $\ 2\ [0;1)$,

$$\mathrm{ES}\ (X) = \frac{1}{1\qquad} \int^{1} \mathrm{VaR}_t(X)\,dt;$$

- Range-Value-at-Risk (RVaR) (Cont-Deguest-Scandolo'10 QF): For $0 \quad < \quad + \qquad 1$,

$$R_{\ ;\ }(X) = \frac{1}{\ } \int^{\ +} \mathrm{VaR}_{1\ \ t}(X)\,dt;$$

Clearly, $R_{0;\ }(X) = \mathrm{ES}\ (X)$ and $\lim_{\ !\ 0} R_{\ ;\ }(X) = \mathrm{VaR}_{1\ \ }(X)$.

Example of premium principles

- Expectation principle:

$$\pi(X) = (1+\theta)\mathbb{E}(X)$$

for $X \in \mathcal{X}$ with $\theta > 0$;

- Standard deviation principle:

$$\pi(X) = \mathbb{E}(X) + \beta\sqrt{Var(X)}$$

for $X \in \mathcal{X}$ with $\beta > 0$;

- Wang's principle:

$$\pi_g(X) = \int_0^1 g(\mathbb{P}(X > x))\,dx$$

for $X \in \mathcal{X}_g$, where $g : [0;1] \to [0;1]$ with $g(0) = 0$ and $g(1) = 1$, and g is increasing.

Book: Young' 04, eleven widely used premium principles.

Loss function

Both f and R_f are non-negative and increasing on $[0, \infty)$ $\Longrightarrow$ Lipschitz-continuous, i.e.,

$$0 \leq f(y) - f(x) \leq y - x, \; 0 \leq x \leq y; \; 0 \leq f(x) \leq x, \; x \geq 0.$$

Examples:

- Quota-share: $f(x) = ax$ with $0 \leq a \leq 1$;
- Stop-loss: $f(x) = (x - c)_+$ with $c > 0$;
- Limited stop-loss:
  $f(x) = (x - a)_+ - (x - b)_+ = \min((x - a)_+ ; b - a)$ with $0 \leq a \leq b$.

(Cai-Chi'20 STRF: review)

Multiple risks

An insurance company usually has many lines of business and each line generates a risk X_i.
Life insurance and non-life insurance.

- Reinsurance for each business X_i: $\quad f_i(X_i) + \quad {}_i(f_i(X_i))$
- The total risk: $S^{\mathbf{f}} = \sum_{i=1}^{n} X_i \quad f_i(X_i) + \quad {}_i(f_i(X_i))$, where $\mathbf{f} = (f_1, \ldots, f_n)$
- The task isto minimize $\quad (S^{\mathbf{f}})$.

Literature

- Cai-Wei'12 IME: $(X) = E(u(X))$, $_i(X) = (1 + \ _i)E(X)$, and $(X_1; : : : ; X_n)$ are positive dependence through stochastic ordering
- Cheung-Sung-Yam'14 JRI: : convex risk measure, $_i(X) = (1 + \ _i)E(X)$, $(X_1; : : : ; X_n)$ are comonotonic (the worst case scenario)
- Bernard-Liu-Vandu el'20 JEBO: $(X) = E(u(X))$, general premium principle, and some speci c dependence structure, $f_i(x) = a_i x$ Quota-Share policy

Conditions on premium principle

We impose the following conditions on $_i$:

(i) Distribution invariance: For $Y, Z \in X$, $_i(Y) = ($

Limited stop loss policy:

Theorem

For $n = 2$, suppose that $F_1^{\,1}$ and $F_2^{\,1}$ are continuous over $(0;1)$, then

$$\inf_{(f_1;f_2)\in D^2} \; \sup_{(X_1;X_2)\in E_2(F)} \mathrm{VaR} \left(S_2^f(X_1;X_2)\right) = \inf_{(a_1;a_2;b_1;b_2)\in A(P)} \; \inf_{t\in[0;1]} L_1(a_1;a_2;b_1;b_2;t);$$

where

$$L_1(a_1;a_2;b_1;b_2;t) = \mathrm{VaR}_{+t}(X_1 \; I_{a_1;b_1}(X_1)) + \mathrm{VaR}_{1 \; t};$$

Theorem

Suppose $F_1^{\ 1}(\ );:::;F_n^{\ 1}(\ )$ are all continuous over $(0;1)$ and $\ 2\ (0;1)$. If each of $F_1;:::;F_n$ is convex beyond its -quantile, then

$$\inf_{f2D_1^n} \ \sup_{(X_1;:::;X_n)2E_n(F)} \mathrm{VaR}\ (S_n^f(X_1;:::;X_n))$$
$$= \inf_{(a;b;c;d)2A_1} \ \inf_{2(1\ \ )_n} H(a;b;c;d;\ );$$

where

$$H(a;b;c;d;\ ) = \textstyle\sum_{i=1}^n f\,R_{\ i;\ 0}(X_i) \quad R_{\ i;\ 0}(h_{a_i;b_i;c_i;d_i}(X_i)) + \ _i(h_{a_i;b_i;c_i;d_i}(X_i))\,g.$$

Additionally, if $\ _i$ are continuous, $(h_{a_1;b_1;c_1;d_1};:::;h_{a_n;b_n;c_n;d_n})$ is the optimal ceded loss functions for the worst case scenario provided

$$(a;b;c;d) = \arg \inf_{(a;b;c;d)2A_1} \ \inf_{2(1\ \ )_n} H(a;b;c;d;\ )$$

Concave distributions on tail part

To guarantee that $X \mapsto f(X)$ has a concave distribution on its tail part,

$$D_2^n = \{ f \mid f = (f_1, \ldots, f_n) : f \in D,\ f_i \text{ is concave for } i = 1, \ldots, n \}$$

$$g_{a,b}(x) := a \min(x, b) - a$$

Concave tail distributions

Theorem

Suppose $F_1^{\ 1}(\ ); \dots; F_n^{\ 1}(\ )$ are all continuous over $(0; 1)$ and $\ 2 (0; 1)$. If each of $F_1; \dots; F_n$ is concave beyond its -quantile, then

$$\inf_{f \in D_2^n} \sup_{(X_1;\dots;X_n) \in E_n(F)} \mathrm{VaR}\ (S_n^f(X_1; \dots; X_n))$$

$$= \inf_{(a;b) \in A_2} \inf_{\ 2 (1\ \ )_n} G(a; b;\ );$$

where $G(a; b;\ ) = \sum_{i=1}^{n} \{ R_{\ i;\ 0}(X_i) \quad R_{\ i;\ 0}(g_{a_i;b_i}(X_i)) + \quad {}_i(g_{a_i;b_i}(X_i)) \}$.

Additionally, if $_i$ are continuous, $(g_{a_1;b_1}; \dots; g_{a_n;b_n})$ is the optimal ceded loss functions for the worst case scenario provided

$$(a; b) = \arg \inf_{(a;b) \in A_2} \inf_{\ 2 (1\ \ )_n} G(a; b;\ )$$

RVaR

- We extend the result (Theorem 1) of Blanchet-Lam-Liu-Wang 20' on convolution bounds on RVaR aggregation from marginal with decreasing densities in the tail part to those with concave distribution in the tail part.
- We obtain similar results on the optimal reinsurance problems.

Numerical study $n = 2$

We solve

$$\min_{f \in D_2} \; \max_{(X_1, X_2) \in E_2} \mathrm{VaR}\left(S_2^f(X_1; X_2) \right) ; \qquad (1)$$

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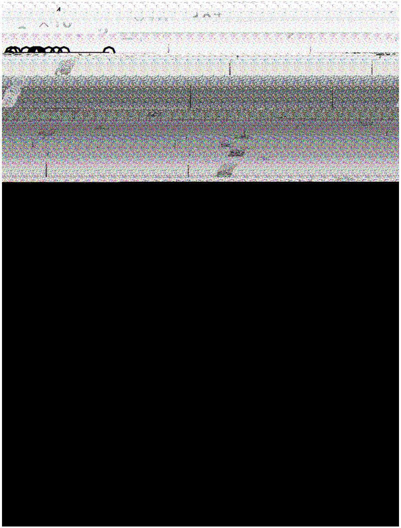
Example: Exponential marginals

- X_i $\quad Exp(\ _i)$ with $E(X_i) = \ _i > 0$
- $_1 = 8000$, $\ _2 = 3000$, $\ _1 = 0{:}8$, $\ _2 = 0{:}3$, $\ = 0{:}95$ and $n = 200$



Summary

- Our main results show that finding the optimal ceded loss functions for the worst case reinsurance models with dependence uncertainty boils down to finding the minimiser of a deterministic function.

Thank You!