An axiomatic theory for comonotonicity-based risk sharing

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1. Introduction

Consider a pool of individual random future losses.

Decentralized risk-sharing:

- Refers to risk-sharing (RS) mechanisms under which the participants in the pool share their risks among each other.
- Each participant in the risk-sharing pool is compensated *ex-post* from the pool for his loss.
- In return, each participant pays an ex-post contribution to the pool.
- These contributions follow from a risk-sharing rule, satisfying the *self-financing condition*.

Decentralized risk-sharing does not require an insurer, but an administrator.

2. Risk-sharing and risk-sharing rules

Agents and their losses

Let χ be an appropriate (su¢ ciently rich) set of r.v.'s in the probability space $(W, \quad , P)$, representing random losses at time 1.[1]

Consider n economic agents, numbered $i = 1, 2, \ldots, n$.

Each agent i faces a loss X_i χ at the end of the observation period $[0, 1]$.

Without insurance or pooling, each agent bears his own loss:

2. Risk-sharing and risk-sharing rules

Pools of losses

The joint cdf of the loss vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is denoted by $F_{\mathbf{X}}$.

The marginal cdf's of the individual losses are denoted by $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$.

The aggregate loss faced by the n agents with loss vector $\mathbf{X}$ is denoted by $S_{\mathbf{X}} = \sum_{i=1}^{n} X_i$.

Hereafter, we will often call $\mathbf{X}$ the pool, and call each agent a participant in the pool.

<u>Definition</u>: For any pool X χ^n with aggregate loss S_X the set $_n(S_X)$ is defined by:

$$_n(S_X) = \left\{ (Y_1, Y_2, \ldots, Y_n) \quad \chi^n \quad \sum_{i}^{n} Y_i = S_X \right\}$$

The elements of $_n(S_X)$ are called the n-dimensional <u>allocations</u> of S_X in χ^n.

Definition: Risk-sharing in a pool $\mathbf{X} \quad \chi^n$ is a two-stage process.

- Ex-ante step (at time 0):
  The losses X_i in the pool are re-allocated by transforming $\mathbf{X}$ into another random vector $\mathbf{H}_{\mathbf{X}} \quad {}_n(S_{\mathbf{X}})$:

$$\mathbf{H}_{\mathbf{X}} = \left(H_{\mathbf{X},1}, H_{\mathbf{X},2}, \ldots, H_{\mathbf{X},n}\right)$$

- Ex-post step (at time 1):
  - Each participant i *receives from the pool* the realization of his loss X_i.
  - In return, he *pays to the pool* a contribution equal to the realization of his re-allocated loss $H_{\mathbf{X},i}$.

Remark: As $\mathbf{H}_{\mathbf{X}} \quad {}_n(S_{\mathbf{X}})$, risk sharing is self-financing:

$$\sum_{i=1}^{n} H_{\mathbf{X},i} = \sum_{i=1}^{n} X_i$$

Definition: A risk-sharing rule is a mapping $\mathrm{H} : \chi^n \qquad \chi^n$ satisfying the self-financing condition:

$$\mathrm{X} \qquad \mathrm{H}_{\mathrm{X}} \qquad {}_{n}(S_{\mathrm{X}}), \qquad \text{for any } \mathrm{X} \quad \chi^n$$

Remarks: For any participant i in the pool $\mathrm{X} = (X_1, \ldots, X_n)$,

X_i is called his loss, (paid by the pool).

$H_{\mathrm{X},i}$ is called his contribution, (paid to the pool).

Contribution vector:

$$\mathrm{H}_{\mathrm{X}} = (H_{\mathrm{X},1}, H_{\mathrm{X},2}, \ldots, H_{\mathrm{X},n})$$

Notation: (χ^n) is the class of all cdf 's of elements in χ^n.

Definition: $H : \chi^n \quad \chi^n$ is an internal RS rule if there exists a function $h : R^n \quad (\chi^n) \quad R^n$ such that the contribution vector H_X of any $X \quad \chi^n$ is given by :

$$H_X = h\left(X, F_X\right)$$

Remarks:

h is called an internal function of the RS rule H.

Hereafter, we only consider internal risk-sharing rules.

De...nition: $\mathrm{H} : \chi^n \quad \chi^n$ is an aggregate RS rule if there exists a function $\mathrm{h}^{\mathrm{aggr}} : \mathrm{R} \quad (\chi^n) \quad \mathrm{R}^n$ such that the contribution vector H_X of any $\mathrm{X} \quad \chi^n$ is given by:

$$\mathrm{H}_\mathrm{X} = \mathrm{h}^{\mathrm{aggr}}\left(S_\mathrm{X}, F_\mathrm{X}\right)$$

Property: Any aggregate RS rule H is internal, with internal function h satisfying:

$$\mathrm{h}\left(\mathrm{X}; F_\mathrm{X}\right) = \mathrm{h}^{\mathrm{aggr}}\left(S_\mathrm{X}, F_\mathrm{X}\right) \qquad \text{for any } \mathrm{X} \quad \chi^n$$

Definition: $\mathrm{H} : \chi^n \quad \chi^n$ is a dependence-free RS rule if there exists a function $\mathrm{h}^{\text{dep-free}} : \mathrm{R}^n \quad (\quad (\chi))^n \quad \mathrm{R}^n$ such that the contribution vector H_{X} of any $\mathrm{X} \quad \chi^n$ is given by:

$$\mathrm{H}_{\mathrm{X}} = \mathrm{h}^{\text{dep-free}}\left(\mathrm{X}, F_{X_1}, \ldots, F_{X_n}\right)$$

Property: Any dependence-free RS rule H is internal, with internal function h satisfying:

$$\mathrm{h}\left(\mathrm{X}; F_{\mathrm{X}}\right) = \mathrm{h}^{\text{dep-free}}\left(\mathrm{X}, F_{X_1}, \ldots, F_{X_n}\right) \qquad \text{for any } \mathrm{X} \quad \chi^n$$

De…nition[2]: The conditional mean RS rule H^{cm} is de…ned by

$$H_i^{\mathrm{cm}}(\mathrm{X}) = \mathrm{E}\left[X_i \quad S_{\mathrm{X}}\right], \qquad i = 1, 2, \ldots, n,$$

for any $\mathrm{X} \quad \chi^n$.

Interpretation: Each participant contributes the expected value of the loss he brings to the pool, given the aggregate loss experienced by the pool.

Property:

H^{cm} is internal and aggregate, but not dependence-free.

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4. The quantile risk-sharing rule

Motivation

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be the realization of $\mathbf{X}$.

There exist probabilities $p_1, \ldots, p_n$ such that

$$\mathbf{x} = \Big(F_{X_1}^{1},$$

4. The quantile risk-sharing rule

De…nition:

Under the quantile RS rule $\mathrm{H}^{\text{quant}} : \chi^n \qquad \chi^n$, the contribution vector of $\mathrm{X} \quad \chi^n$ is given by

$$\mathrm{H}_{\mathrm{X}}^{\text{quant}} = \mathrm{h}^{\text{quant}}\left(S_{\mathrm{X}}, F_{\mathrm{X}}\right)$$

where $\mathrm{h}^{\text{quant}} : \mathrm{R} \qquad (\chi^n) \qquad \mathrm{R}^n$ is de…ned by

$$h_i^{\text{quant}}\left(s, F_{\mathrm{X}}\right) = F_{X_i}^{\;\,1}\left(F_{S_{\mathrm{X}}^c}(s)\right), \qquad i = 1, 2, \ldots, n$$

Properties:

$\mathrm{H}^{\text{quant}}$ satis…es the self-…nancing condition.

$\mathrm{H}^{\text{quant}}$ is an aggregate RS rule.

$\mathrm{H}^{\text{quant}}$ is a dependence-free RS rule.

5. The 'stand-alone for comonotonic pools' property

De…nition: X χ^n is a comonotonic pool in case

$$X \stackrel{d}{=} \left(F_{X_1}{}^{1}(U), \ldots, F_{X_n}{}^{1}(U)\right)$$

De…nition: A RS rule H : χ^n χ^n satis…es the stand-alone for comonotonic pools property if for any comonotonic pool X^c χ^n, one has that

$$H_{X^c} = X^c$$

Property: H^{quant} satis…es the stand-alone for comonotonic pools property.

6. Axiomatic characterization of the quantile RS rule

Theorem:

Consider the internal RS rule $H : \chi^n \to \chi^n$.

H is the quantile RS rule if, and only if, it satisfies the following axioms:

(1) H is aggregate.

(2) H is dependence-free.

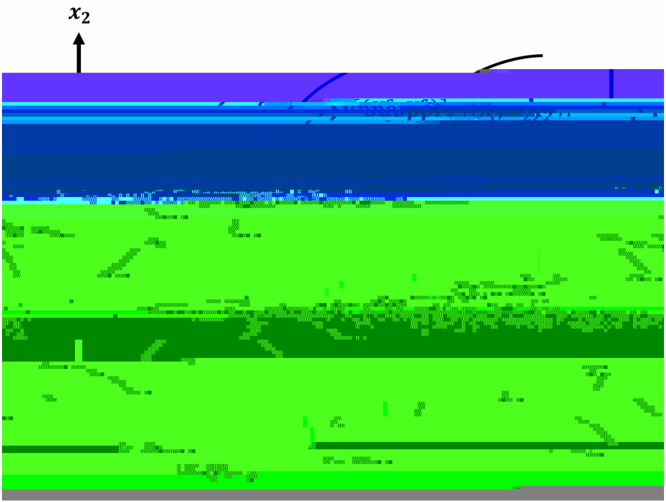
(3) H is (generalized) stand-alone for comonotonic pools[4].

Proposition:
The axioms (1), (2) and (3) are independent.

[4] The 'generalized stand-alone for comonotonic pools' property is a slightly stronger property than the 'stand-alone for comonotonic pools' property, see D,R,C,D (2023).

6. Axiomatic characterization of the quantile RS rule

Graphical proof of the theorem (bivariate case)



$$h\left(x\ ,F_X\right)\overset{\text{axiom 1}}{=}h\left(x^c,F_X\right)\overset{\text{axiom 2}}{=}h\left(x^c,F_{X^c}\right)\overset{\text{axiom 3}}{=}x^c$$

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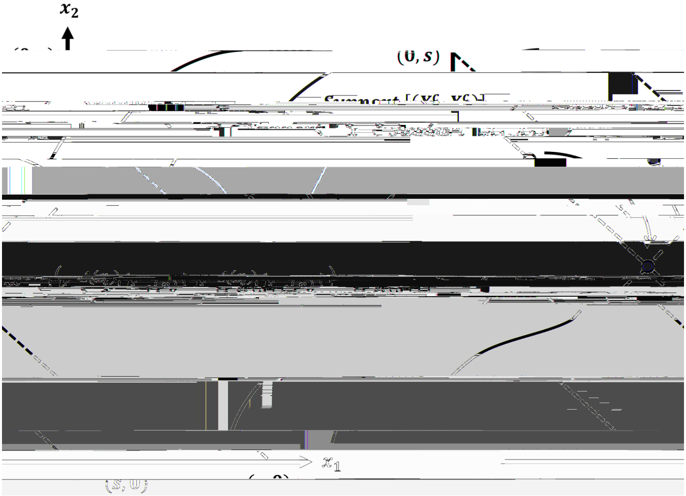

$$h\left(x\;, F_X\right) \overset{\text{axiom 1}}{=} h\left(x^c, F_X\right) \overset{\text{axiom 2}}{=} h\left(x^c, F_{X^c}\right) \overset{\text{axiom 3}}{=} x^c$$

7. Example of a non-internal risk-sharing rule

Consider the RS rule H : χ^n χ^n, where any X χ^n is a pool of health-related costs of the participants.

References

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