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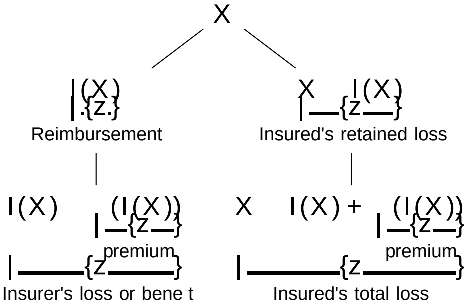
Insurance 101

Insurance is an effective risk management tool used to protect against contingent losses of market participants.



where $I \in \mathcal{I}$ is an admissible indemnity function, and is a premium principle.

Classical optimization problems in insurance

Popular optimal (re-)insurance design problems:

1. Maximize expected utility:

$$\max_{I \in \mathcal{I}} E\left[v(w - X + I(X) - \pi(I(X)))\right].$$

   Arrow (1963): optimality of a stop-loss contract.
   Gerber(1979), Young (1999), Kaluszka (2001,2005), etc.

2. Minimize risk measure:

$$\min_{I \in \mathcal{I}} \rho(X - I(X) + \pi(I(X))).$$

   Cai et al. (2008), Kaluszka and Okolewki (2008), Bernard and Tian (2009), Cheung (2010), etc.

All problems are considered under the assumption that the distribution of X is known. Can we take this assumption for granted?

Uncertainty

From data to models

- Parameter uncertainty
  Estimation error, simulation error, etc
- Model uncertainty
  Choice of models, complexity of models, etc.

Distributional uncertainty

- Only partial information about the true distribution are observed from the historical data.
- Changes of the underlying risks
- In a conservative decision, the worst-case distribution is important

Worst-case scenario

Suppose an agent faces an underlying risk X

- ℓ is the loss function/strategy the agent adopts.
- ρ is the risk measure used to quantify the agent's risk exposure
- $\mathcal{S}$ is the uncertainty set includes all distributions of alternative risks considered

From the perspective of risk management, the worst-case scenario in which the agent has the largest risk exposure is of special interests.

The agent's optimization problem with model uncertainty can be formulated as

$$\min_{\ell} \underbrace{\sup_{F \in \mathcal{S}} \rho(\ell(X^F))}_{\text{worst-case scenario}}; \quad X^F \sim F.$$

Literature

In the literature of insurance

Asimit et al. (2017): for $= \mathrm{VaR}; \mathrm{ES}$,

$$\begin{cases} \min_{(I;P) \in I} \max_{R\, k \in M} \{ {}_{P_k}(X \quad I(X) + P) \}; \\ \text{s.t. } !\ {}_0 + (1 + \quad) H_{P_k}(I(X)) \quad P \quad P; \forall k \in M \end{cases}$$

where P_k, $k \in M$ includes finite many probability measures.

Birghila and Pflug (2019)

$$\min_{I \in I} \max_{F \in C} \{ (X^F \quad I(X^F) + \quad (I(X^F)) \}; \quad \text{s.t.} \quad (I(X^F)) \quad B$$

where C is the convex cone of n reference distributions.

Liu and Mao (2021): for $= \mathrm{VaR}; \mathrm{ES}$,

$$\min_{d \quad 0} \sup_{F \in S(\ ;\ )} (X^F \wedge d + (1 + \quad) E^F[(X^F \quad d)_+]):$$

where $S(\ ;\ )$ gives first & second moments constraints.

In this talk, we focus on the worst-case scenario for an agent

$$\sup_{F \in S} \rho_h(\ell(X^F)), \quad X^F \sim F$$

where

- ρ_h is a distortion risk measure (e.g. Dhaene et al. (2012)):

$$\rho_h(X^F) = -\int_{-\infty}^{0} h(F(x))\,dx + \int_{0}^{\infty} 1 - h(F(x))\,dx = \int_{0}^{1} \phi(u) F^{-1}(u)\,du,$$

  where $h : [0,1] \mapsto [0,1]$ is non-decreasing (convex) with $h(0) = 0$ and $h(1) = 1$, and $\phi(u) = h'(u)$, $0 < u < 1$

- S is the uncertainty set defined by Wasserstein distance constraints
- ℓ is the loss function/strategy the agent adopts.

Uncertainty set with Wasserstein distance constraint

For X   F and Y   G, for k   1, the Wasserstein distance is

$$W_k(X;Y) = W_k(F;G) = \left(\int_0^1 F^{-1}(x) - G^{-1}(x)^{k}\right)^{1/k}:$$

The uncertainty set with Wasserstein distance constraint

S = f r. v. Y : $W_k(Y;X)$ "W97 0 87luel)

Uncertainty set with Wasserstein distance constraint

Theorem (Proposition 4 in Liu et al. (2022))

For a continuous and convex distortion function **h**,

$$\sup \ {}_h(X^G) : W_k(G;F) \quad \varepsilon \ = \ {}_h(X^F) + \varepsilon\| \ \|_q;$$

where $q = (1 \quad 1/k) \ ^1$ with the convention $0 \ ^1 = 1$, and $|| \ ||_q$ is the L_q-norm.

For $k > 1$, the above maximum value is attained by the worst-case distribution

$$G \ ^1(t) = F \ ^1(t) + \varepsilon\frac{(\ (t))^{q \ 1}}{\| \ \|_q^{q/k}}; \qquad 0 < t < \ 1:$$

Example { Expected shortfall (ES)

Take = ES for 2 $(0;1)$, then $(X) = \int_0^1 \mathrm{VaR}_t(X)\,dh(t)$, where

$$h(t) = \frac{1}{1}(t \qquad)^+ \qquad \text{and} \qquad (t) = \frac{1}{1} 1_{[\;;\;1]}:$$

The worst-case value is

$$\sup \quad \overset{n}{} \; \mathrm{ES}\ (X^{G}) : W_k(G;F$$

Uncertainty set with Wasserstein distance constraint

Uncertainty set is

$$S = \{G : W_k(G, F) \leq \varepsilon\}$$

where $X^F \sim F$ is considered as a reference distribution, and ε is the tolerant bound for the Wasserstein distance.

Consider the worst-case scenario:

$$\sup_{G \in S} \rho_h(\ell(X^G)) = \sup \{\rho_h(\ell(X^G)) ; W_k(G, F) \leq \varepsilon\};$$

with two types of loss functions:

- Stop-loss function: (optimal to the utility maximization)

$$\ell(x) = (x - d)^+$$

- Limited-loss function: (optimal to the VaR minimization)

$$\ell(x) = \min\{x, M\}$$

Stop-loss function

Take $\ell_1(x) = (x - d)^+$ for $d > \text{ess-inf}(X)$

Worst-case risk measure

$$\sup \left\{ \rho_h((X^G - d)^+) : W_k(G, F) \leq \varepsilon \right\}$$

For $\lambda \in [0, 1]$, define $\gamma_{1;\lambda} := \gamma I_{[\lambda;1]}$ which is again a non-negative and increasing function.

$$\begin{aligned}
\sup_{G \in S} \rho_h\left((X^G - d)^+\right) &= \sup_{G \in S} \int_{G(0)}^{1} \gamma(u) \left(G^{-1}(u) - d\right) du \\
&= \sup_{G \in S} \max_{\lambda \in [0,1]} \int_{\lambda}^{1} \gamma(u) \left(G^{-1}(u) - d\right) du \\
&= \sup_{\lambda \in [0,1]} \underbrace{\sup_{G \in S} \int_{0}^{1} \gamma_{1;\lambda}(u) \left(G^{-1}(u) - d\right) du}_{\text{worst-case without transform}};
\end{aligned}$$

Wasserstein distance constraint and stop-loss transform

Theorem (Cai et al. (2022b))

Take $k \quad 1$ and $q = (1 \quad 1{=}k)^{\ 1}$.

(i) The worst-case risk measures value is

$$\sup \ \overset{n}{\ } \ {}_h((X^G \quad d)^+) : W_k(G;F) \quad " \ \overset{o}{\ }$$

$$= \max_{2[0;1]} \int_0^1 {}_{1;}\ (u)F^{\ 1}(u)du + "k_{\ 1;}\ k_q \quad dk_{\ 1;}\ k_1 \quad :$$

(ii) The worst-case distribution is given by

$$G^{\ 1}(t) = F^{\ 1}(t) + " \ \frac{(\ {}_{1;}\ \ (t))^{q \ \ 1}}{k_{\ 1;}\ \ k_q^{q=k}}; \quad 0 < t < \ 1:$$

where $\quad$ is the maximizer in (i).

Example - Expected shortfall

Take $= \mathrm{ES}$ for some $2\ (0;1)$.

(i) The worst-case value is

$$\sup \left\{ \mathrm{ES}\ ((X^{G}\quad d)^{+}) : W_k(G;F) \quad " \right\}$$

$$= \frac{1}{1} \max_{2[\ ;\ 1]} \left\{ (1 \quad)\ \mathrm{ES}\ (X^{\hat{F}}) \quad d \ + "(1 \quad)^{1=k} \right\} :$$

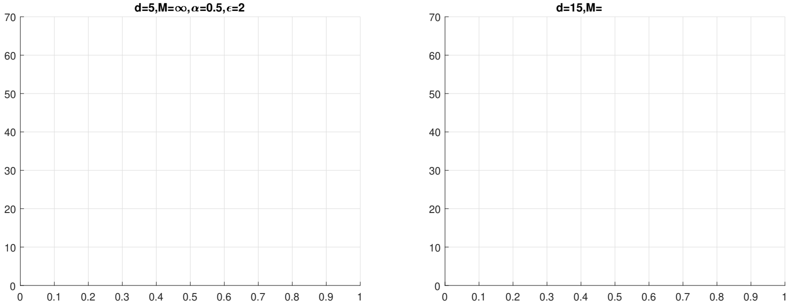
(ii) The worst-case distribution is

$$G \ {}^{1}(t) = F \ {}^{1}(t) + " \ \frac{(\ _{1;}\ \ (t))^{q \ \ 1}}{k\ \ _{1;}\ \ k_q^{q=k}}$$

where $_{1;} \ = \frac{1}{1} I_{[\ \_\ \ ;1]}$ and is the solution to the maximization problem in (i).

Example - Wang's premium

Figure: Worst-case distributions with stop-loss function.



Limited-loss function

Take $\ell_2(x$

Wasserstein distance constraint and limited-loss transform

Theorem (Cai et al. (2022b))

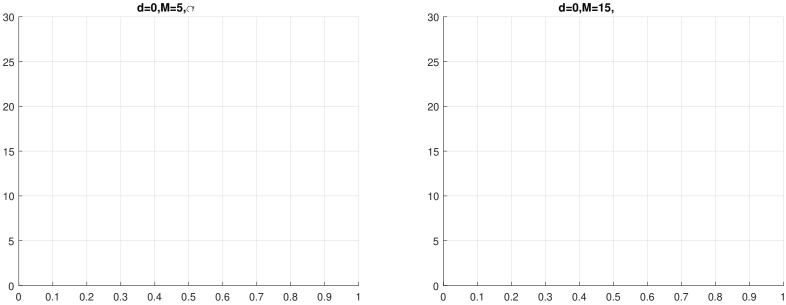
Let $k = 2$. The worst-case distribution is given by

$$(F\ )^{1}(u) = \begin{cases} F^{1}(u) + \quad (u); & \text{for } 0 < u \quad ; \\ M; & \text{for } \quad < u \quad F(M); \\ F^{1}(u); & \text{for } F(M) < u < 1 \end{cases}$$

where $\ > 0$ and $\ 2\ (0; F(M))$ satis es $W_2(F\ ; F) = "$.

Example - Wang's premium (cont')

Figure: Worst-case distributions with limited loss function.



Wasserstein distance constraint and limited stop-loss transform

- Wang's premium $_h$ with $h(u) = 1 \quad (\quad ^{1}(1 \quad u) + 0.5)$.
- Exponential reference $F_1(x) = 1 \quad e^{\ x=4}$, $x \quad 0$
- Pareto reference $F_2(x) = 1 \quad \frac{12}{x+12}^{\ 4}$
- Limited stop-loss function

$$`(x) = \max \ (x \quad d)^+ ; M$$

- Wang's premium in the worst-case:

$$\sup \quad _h \ \max \ (X^G \quad d)^+ ; M \quad ; W_2(G; F_i) \quad " \quad ; \quad i = 1; 2:$$

Isotonic Projection: For $h \in L^2(0;1)$, let

$$h^{\uparrow} = \arg\min_{k \in K} \| h \quad k \|^2;$$

$$\text{where } K = \quad k : (0;1) \mapsto \mathbb{R} \quad \int_0^1 k(u)^2 du < \infty \ ; k \text{ non-decreasing} \quad :$$

Notation

- Denote $_{1;}\ (u) := \quad (u)\, 1_{[\ ;\ 1]}(u)$, for $u \in [0;1]$, and the isotonic Projection for $_{1;} \ + \ F^{\ 1}$ for some $\quad 0$ as

$$h^{\uparrow}_{1;\ ;} = \arg\min_{h \in K} \| h \quad _{1;} \quad F^{\ 1} \|_2 :$$

- Denote $_{2;}\ (u) := \quad (u)\, 1_{[0;\ ]}(u)$, for $u \in [0;1]$, and the isotonic Projection for $_{2;} \ + \ F^{\ 1}$ for some $\quad 0$ as

$$h^{\uparrow}_{2;\ ;} = \arg\min_{h \in K} \| h \quad _{2;} \quad F^{\ 1} \|_2 :$$

Wasserstein distance plus moments constraints and stop-loss transform

Theorem (Cai et al. (2022a))

Consider the worst-case problem $\sup_{G \in S} h\,(Y^G \;\; d)_+ \;:$
The quantile function of the worst-case distribution is

$$G^{\;1}(u) = \quad + \quad \left(\frac{h^{"}_{1;\;\;;}(u) \quad a_{\;;}}{b_{\;;}} \right) ; \qquad 0 < u < 1;$$

where $a_{\;;} = E[h^{"}_{1;\;\;;}(U)]$, $b_{\;;} = \sqrt{var(h^{"}_{1;\;\;;}(U))}$, $\;\; > \;\; 0$ is determined uniquely by the distance constraint $W_2(F;G\;\;) = \;"$, and

$$= \arg\max_{\;2\,[0;1]} \int_0^1 \;_{1;}\,(u) \;\; G^{\;1}(u) \quad d \;\; du:$$

Example { Expected shortfall

Assume the reference distribution is $F(x) = 1 \quad e^{x=5}$, $= \quad = 5$, $" = 1$, and $_h = \mathrm{ES}_{0:9}$.ssume3. Wors9c28.909 1(0)]14R8= 1

Wasserstein distance plus moments constraints and limited-loss transform

Theorem (Cai et al. (2022a))

Consider the worst-case problem $\sup_{G \in S} h \; Y^{G} \wedge M \;:$

The quantile function of the worst-case distribution is

Example { Expected shortfall

Assume the reference distribution is $F(x) = 1 \quad e^{\ x=5}$, $\ = \ = 5$, $" = 1$, and $_h = ES_{0:9}$:

d	
10	[0; 0:9]
20	0:9835

Summary

- In this talk we discuss multiple model uncertainty models
  - Distortion risk measure
  - With or without transform
    - Stop-loss, limited-loss
  - Wasserstein distance, moments contraints

- Future works
  - Other risk measures
  - General transformation
  - Various uncertainty sets: likelihood ratio, KL-divergent, etc.
  - Novel techniques to characterize worst-case distribution and worst-case risk measure value

Reference II

Dhaene, J., Kukush, A., Linders, D., and Tang, Q. (2012). Remarks on quantiles and distortion risk measures. European Actuarial Journal, 2(2):319{328.

El Ghaoui L, Oks M, Oustry F (2003) Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. Oper. Res. 51(4):543{556.

Li, J. Y.-M. (2018). Closed-form solutions for worst-case law invariant risk measures with application to robust portfolio optimization. Operations Research, 66(6):1533{1541.

Liu, H. and Mao, T. (2021). Distributional robust reinsurance with value-at-risk and conditional value-at-risk. Available at SSRN 3849078

Liu, F. and Wang, R. (2021). A theory for measures of tail risk. Mathematics of Operations Research, 46(3), 1109{1128.

Hu, X., Yang, H., and Zhang, L. (2015). Optimal retention for a stop-loss reinsurance with incomplete information. Insurance: Mathematics and Economics, 65:15{21.

Sion, M. (1958). On general minimax theorems, Paci c Journal of mathematics , 8(1), 171{176.